Lesson 34. Double Integrals II
I. Integrating over General Regions
II. Switching Order of Integration

Quiz Friday 12/1/2023
Lessons 33 and 34.

I. Integrating over General Regions

Last Class - I gave you the limits of integration

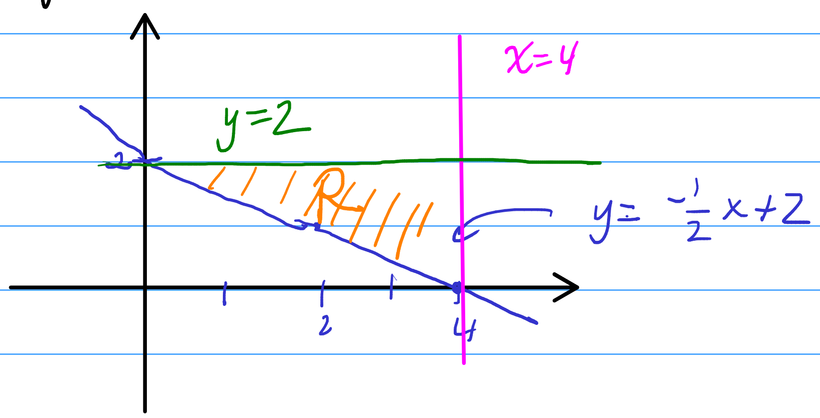
Today's Class - I will give you the region of integration (R). Your first step will be to set up the integrals and their limits

$\boxed{\text{Ex}}$ (Rogawski et.al., §15.2, #5)

Evaluate $\iint_R x^2 y \, dA$

where R is the region bounded by

$y = \frac{-1}{2}x + 2$, $x = 4$, and $y = 2$



Region:
top: $y = 2$
bottom: $y = -\frac{1}{2}x + 2$

right: $x = 4$
left: $x = 0$

$$\iint_R x^2y\,dA = \int_0^4 \int_{-\frac{1}{2}x+2}^{2} x^2 y\, dy\, dx$$

limits on outside integral are #'s.

$$= \int_0^4 \left[x^2 \cdot \frac{y^2}{2} \Big|_{y=-\frac{1}{2}x+2}^{y=2} \right] dx$$

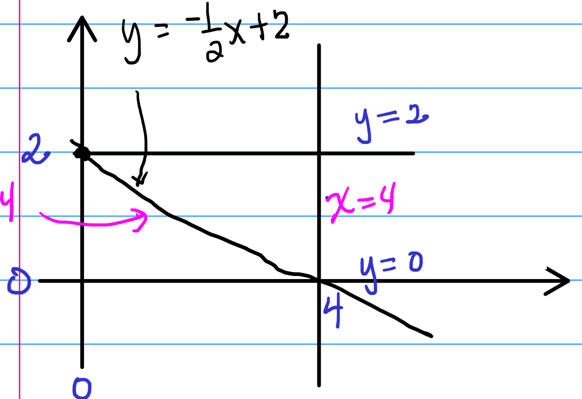
$$\left(\tfrac{-1}{2}x+2\right)\left(\tfrac{-1}{2}x+2\right) = \tfrac{1}{4}x^2 - 2x + 4$$

$$= \int_0^4 \left[x^2\left(\frac{2^2}{2}\right) - \underbrace{x^2\frac{\left(\frac{-1}{2}x+2\right)^2}{2}}_{-\frac{x^2}{2}\left(\frac{1}{4}x^2-2x+4\right)} \right] dx$$

$$= \int_0^4 \left[\cancel{2x^2} - \frac{x^4}{8} + x^3 - \cancel{2x^2} \right] dx$$

$$= -\frac{1}{8}\frac{x^5}{5} + \frac{x^4}{4} \Big|_{x=0}^{x=4} = 38.4$$

You can also do this problem as $\iint_R x^2y\, dx\, dy$



In this view.

top: $y=2$
bottom: $y=0$
#'s b/c dy is outside integral.

right: $x=4$
left: $x=-2y+4$

$$y = \frac{-1}{2}x + 2$$

$$y - 2 = \frac{-1}{2}x$$

$$-2y + 4 = x$$

$$\iint_R x^2y\,dA \overset{\text{is also}}{=} \int_0^2 \int_{x=-2y+4}^{x=4} x^2 y\, dx\, dy$$

$$= \int_0^2 \left[\frac{x^3}{3} y \Big|_{x=-2y+4}^{x=4} \right] dy$$

$(-2y+4)(-2y+4)(-2y+4)$

$(4y^2-16y+16)(-2y+4)$

$-8y^3+48y^2-96y+64$

$$= \int_0^2 \left[\frac{4^3}{3} y - \frac{(-2y+4)^3}{3} y \right] dy$$

$$= \int_0^2 \left[\frac{64}{3} y - \frac{(-8y^3+48y^2-96y+64)\,y}{3} \right] dy$$

$$= \int_0^2 \left[\cancel{\frac{64}{3} y} + \frac{8}{3} y^4 - \frac{\overset{16}{\cancel{48}}}{\cancel{3}} y^3 + \frac{\overset{32}{\cancel{96}}}{\cancel{3}} y^2 - \cancel{\frac{64}{3} y} \right] dy$$

$$= \frac{8}{15} y^5 - 4y^4 + \frac{32}{3} y^3 \Big|_0^2 \qquad = \boxed{138.4}$$

Same answer as before.

This an example of switching the order of integration.

It was not needed for this problem, but can be essential/required for other problems

II. Switching the Order of Integration

$\boxed{\text{Ex}}$ (Rogawski et. al. §15.2 #32)

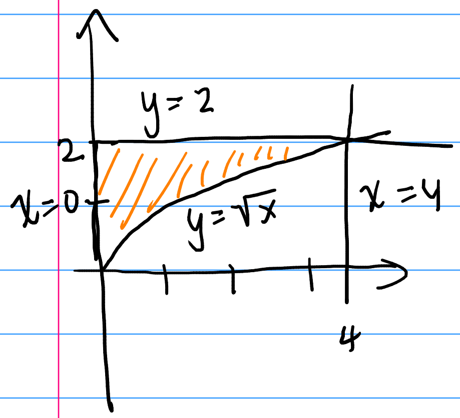
$$\int_0^4 \int_{\sqrt{x}}^2 \sin(y^3)\, dy\, dx$$

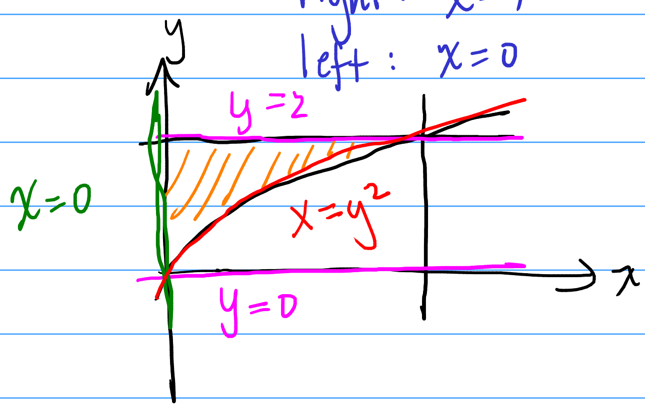
1st step: Find antiderivative $\underbrace{\int \sin(y^3)\, dy}_{\text{we can't do this w/out series}}$.

$$\int_0^4 \int_{\sqrt{x}}^2 \sin(y^3)\, dy\, dx$$

R: top: $y=2$
bottom: $y=\sqrt{x}$

right: $x=4$
left: $x=0$





New perspective: top: $y=2$
bottom: $y=0$

$y=\sqrt{x}$
$y^2=x$

right: $x=y^2$
left: $x=0$

$$\int_0^4 \int_{\sqrt{x}}^2 \sin(y^3)\, dy\, dx = \int_{y=0}^{y=2} \int_{x=0}^{x=y^2} \sin(y^3)\, dx\, dy$$

$$= \int_0^2 \left[\sin(y^3)\, x \Big|_{x=0}^{x=y^2} \right] dy$$

$$= \int_0^2 \left[\sin(y^3)\, y^2 - \sin(y^3)\cdot 0 \right] dy$$

$\frac{1}{3}du$

$$= \int_0^2 \left[y^2 \sin(y^3) \right] dy$$

$u = y^3$
$du = 3y^2\, dy$

$$= \int_0^8 \sin(u) \cdot \frac{1}{3}\, du$$

$\frac{1}{3}du = y^2 dy$

$y=0, \; u=0^3=0$
$y=2, \; u=2^3=8$

$$= -\frac{1}{3}\cos(u) \Big|_0^8$$

$$= -\frac{1}{3}\cos(8) - \left(-\frac{1}{3}\right)\cos(0)^{1} = -\frac{1}{3}\cos(8) + \frac{1}{3}$$

An Answer! ☺

We found it by switching order of integration